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The Local Quantized Superfield, CPT, and the Connection between Spin and Statistics

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Abstract

Starting from the principles of the local quantized ordinary fields in four dimensions the theory is constructed for the local quantized superfield. From the normal connection between between spin and statistics of ordinary fields the local commutativity (anticommutativity) for superfields is obtained. Also the CPT invariance of "Wightman's superfunction" and the general theorem on the connection between spin and statistics are proved.

1. *Introduction*

It is approximately one year since the time when supersymmetry emerged on the physical scene. New fields connected with the supersymmetry algebra, so called "superfields", were defined over the eight-dimensional "extended space-time" (Salam and Strathdee, 1974a) whose points are represented by the pair (x_μ, ϕ_α), where x_μ are space-time coordinates and θ_α is an anticommuting c-number Majorana spinor.

The supersymmetry transformation on this "extended space-time" is given by

$$x_\mu \to x_\mu + \tfrac{1}{2} i\bar{\epsilon}\gamma_\mu\theta$$
$$\theta_\alpha \to \theta_\alpha + \epsilon_\alpha \tag{1.1}$$

where ϵ_α is an anticommuting Majorana spinor. The supersymmetry algebra A is the generalized Lie algebra over the field R of real numbers with "the physical basis" $\{P_\mu, J_{\mu\nu}, S_\alpha\}$ $\mu, \nu = 0, 1, 2, 3, \alpha = 1, 2, 3, 4$, where the usual commutation rules involving P_μ and $J_{\mu\nu}$ must be supplemented by the rules involving S_α:

$$[S_\alpha, P_\mu] = 0$$
$$[S_\alpha, J_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu}S)_\alpha \tag{1.2}$$
$$\{S_\alpha, S_\beta\} = -(\gamma_\mu C)_{\alpha\beta}P_\mu$$

The explanation for why S_α is a Majorana spinor follows from extension of the requirement of CPT invariance of commutation rules of generators of infinitesimal Poincaré transformations to new rules involving S_α(Hrubý, 1974).

For simplicity now we shall work with the real scalar superfield $\phi(x, \theta)$, which can be given by a formal polynomial (for a detailed description see Salam and Strathdee, 1974a) in four anticommuting generators θ_α: $\phi(x, \theta) = \bar{\phi}(x) +$

$$\bar{\phi}^\alpha(x)\theta_\alpha + \tfrac{1}{2}\bar{\phi}^{[\alpha\beta]}(x)\theta_\beta\theta_\alpha + \tfrac{1}{6}\bar{\phi}^{[\alpha\beta\gamma]}(x)\theta_\gamma\theta_\beta\theta_\alpha + \tfrac{1}{24}\bar{\phi}^{[\alpha\beta\gamma\delta]}\theta_\delta\theta_\gamma\theta_\beta\theta_\alpha \tag{1.3}$$

where the coefficients are in general complex valued functions of x_μ and are exactly equivalent to a 16-component set of ordinary fields in four dimensions. Because we shall here assume a real scalar superfield

$$\phi^*(x, \theta) = \phi(x, \theta) \tag{1.4}$$

we shall have Bose components that are real and Fermi components that are Majorana spinors.

Because the ordinary fields in four dimensions are quantized, it is quite natural to raise the question: What is a quantized superfield? The answer to this question is given in section 2. In section 3 "Wightman's superfunctions" are defined and using CPT invariance of ordinary Wightman's functions the CPT invariance of "Wightman's superfunction" is shown. In section 4 we shall see that from the normal connection between spin and statistics of ordinary fields in four dimensions we obtain (a) local commutativity between spinor superfield and scalar superfield, (b) local anticommutativity between spinor superfields, and (c) local commutativity between scalar superfields. On the other hand, starting from the local quantized superfield, we shall prove the normal connection between spin and statistics for superfields.

2. *The Local Quantized Superfield*

(I) The quantized superfield will be the operator valued distribution

$$\phi(f, \theta) = \int \phi(x, \theta)f(x)d^4x \tag{2.1}$$

[where $f \in \mathscr{S}(R_4)$—the space of analytic rapidly decreasing functions], which is acting on Z_2-graded vector space $H = H_0 \oplus H_1$ fulfilling the following conditions:

(1) All operators $\phi(f, \theta)$ have an invariant common dense domain Ω for all $f \in \mathscr{S}(R_4)$ and the vacuum $|0\rangle \in \Omega$, where $\phi(f, \theta)\Omega \subset \Omega$.

(2) $\phi(\alpha f + \beta g, \theta) = \alpha\phi(f, \theta) + \beta\phi(g, \theta), \alpha, \beta \in C$.

(3) $\phi(f, \theta)$ fulfill

if $\psi_1, \psi_2 \in \Omega$, then $\langle\psi_1|\phi(f,\theta)\psi_2\rangle \epsilon \mathscr{S}^*(R_4)$—the space of linear continuous functionals on $\mathscr{S}(R_4)$.

(4)
$$\phi(f, \theta) = \sum_{j=1}^{4} \frac{1}{j!} \bar{\phi}_j^{[\alpha_1 \cdots \alpha_j]} \theta_{\alpha_j} \cdots \theta_{\alpha_1}$$

$$= \sum_{j=1}^{4} \frac{1}{j!} \int \bar{\phi}_j^{[\alpha_1 \cdots \alpha_j]}(x) f(x) d^4x \theta_{\alpha j} \ldots \theta_{\alpha_1} \tag{2.2}$$

This definition of the quantized superfield is more convenient than that of Kotecký (1974), where θ_α are also quantized, because the testing functions have a good physical meaning only in physical space R_4 and not in "extended space-time." Therefore it is physically impossible to define superfunctions as testing functions.

The problem of constructing unitary representations of supersymmetry algebra is solved by Salam and Strathdee (1974b). Then the condition of locally covariant superfield can be formulated as follows:

(II) Let $U(\underset{\sim}{a}, A)$ be a unitary representation of $\mathscr{P}_0 \equiv \mathscr{P}_+^\uparrow$ in H.

Then we shall define the locally covariant superfield:

$$U(\underset{\sim}{a}, A)\phi(f, \theta)U^{-1}(\underset{\sim}{a}, A) = \phi(f_{\{\underset{\sim}{a}, A\}}, V(A)\theta)$$

$$= \sum_{j=1}^{4} \frac{1}{j!} \bar{\phi}^{[\alpha_1 \cdots \alpha_j]}(f_{\{\underset{\sim}{a}, A\}}) V(A)\theta_{\alpha_j} \cdots V(A)\theta_{\alpha_1} \tag{2.3}$$

Also the requirement of local commutativity can be generalized naturally:

(III) Let for the supports f_1 and f_2

$$\operatorname{supp} f_1(x) \cdot f_2(y) \subset \{(x-y)^2 < 0\}$$

be fulfilled; then for operators $\phi(f_1, \theta), \phi(f_2, \theta)$

$$[\phi(f_1, \theta), \phi(f_2, \theta)] = 0 \tag{2.4}$$

holds.

Similarly the requirement of completeness of the theory for normal quantized field can be generalized:

(IV) Vacuum $|0\rangle$ is a cyclic vector under algebra $U \equiv$ {all polynomials $P[\phi(f, \theta)]$}.

So we can see that axioms (I)–(IV) are generalization of axioms required in normal local theory for superfields.

3. *The "Wightman's Superfunctions" and the CPT Invariance*

We can define "Wightman's superfunction" for real scalar superfields as vacuum expectation value of product n superfields:

$$W_{\phi_1(x_1,\theta)\phi_2(x_2,\theta)\cdots\phi_n(x_n\theta)} = \langle 0|\phi_1(x_1,\theta)\phi_2(x_2,\theta)\cdots\phi_n(x_n,\theta)|0\rangle \quad (3.1)$$

This "Wightman superfunction" has its specific characteristic:

(1) The product of n superfields is again a superfield and therefore can be expanded in Taylor's expansion in θ_α, which is finished for $n = 4$ because the condition is fulfilled that the product

$$\theta_{\alpha_1}\theta_{\alpha_2}\cdots\theta_{\alpha_n}$$

vanish for $n > 4$.

(2) Because the vacuum expectation value of odd-numbered spinor fields is equal to zero, all members of (3.1) with odd-numbered anticommuting Majorana spinor fields are equal to zero.

Now we shall do it more precisely. We shall use the expansion (1.3) in the form (Salam and Strathdee, 1974c)

$$\begin{aligned}\phi(x,\theta) &= A(x)\\ &+ \bar\theta\psi(x)\\ &+ \tfrac{1}{4}\bar\theta\theta F(x) + \tfrac{1}{4}\bar\theta\gamma_5\theta G(x) + \tfrac{1}{4}\bar\theta i\gamma_\nu\gamma_5\theta A_\nu(x)\\ &+ \tfrac{1}{4}\bar\theta\theta\bar\theta\chi(x)\\ &+ \tfrac{1}{32}(\bar\theta\theta)^2 D(x)\end{aligned} \quad (3.2)$$

where the coefficients A, F, G, A_ν, and D are Bose fields and ψ, χ are Fermi fields. We start with

$$\begin{aligned}W_{\phi_1(x_1,\theta)\phi_2,\theta)} &\equiv \langle 0|\phi(1,\theta)\phi(2,\theta)|0\rangle\\ &= \langle 0|A(1)A(2)|0\rangle\\ &\quad + \tfrac{1}{4}\bar\theta\theta\langle 0|A(1)F(2) - \bar\psi(1)\psi(2) + F(1)A(2)|0\rangle\\ &\quad + \tfrac{1}{4}\bar\theta\gamma_5\theta\langle 0|A(1)G(2) + \bar\psi(1)\gamma_5\psi(2) + G(1)A(2)|0\rangle\\ &\quad + \tfrac{1}{4}\bar\theta i\gamma_\nu\gamma_5\theta\langle 0|A(1)A_\nu(2) + \bar\psi(1)i\gamma_\nu\gamma_5\psi(2) + A_\nu(1)A(2)|0\rangle\\ &\quad + \tfrac{1}{4}(\bar\theta\theta)^2\langle 0|A(1)D(2) + 2[F(1)F(2) + G(1)G(2) + A_\nu(1)A_\nu(2)]\\ &\quad + D(1)A(2) - 2[\bar\psi(1)\chi(2) + \bar\chi(1)\psi(2)]|0\rangle\end{aligned} \quad (3.3)$$

So we can generalize for n superfields:

$$\begin{aligned}W_{\phi_1(x_1,\theta)\cdots\phi_n(x_n,\theta)} &= \langle 0|A(1)A(2)\cdots A(n)|0\rangle\\ &+ \tfrac{1}{4}\bar\theta\theta\,\overline{W}^{\text{scalar (II)}}_{1,\ldots,n}\\ &+ \tfrac{1}{4}\bar\theta\gamma_5\theta\,\overline{W}^{\text{pseudoscalar (II)}}_{1,\ldots,n}\\ &+ \tfrac{1}{4}\bar\theta i\gamma_\nu\gamma_5\theta\,\overline{W}^{\text{axial vector (II)}}_{1,\ldots,n}\\ &+ \tfrac{1}{32}(\bar\theta\theta)^2\,\overline{W}^{\text{scalar (IV)}}_{1,\ldots,n}\end{aligned} \quad (3.4)$$

The symbols $\overline{W}$ are as follows:

(a) $\overline{W}^{\text{scalar (II)}}_{1,\ldots,n}$: the vacuum expectation value from the linear combination of all products of n coefficients of expansion (3.2) with no more than *two* Majorana spinors θ_α and with the whole linear combination transforming as a scalar [for example $\overline{W}^{\text{scalar (II)}}_{1,2} = \langle 0|A(1)F(2) - \bar{\psi}(1)\psi(2) + F(1)A(2)|0\rangle$]. $\overline{W}^{\text{pseudoscalar (II)}}_{1,\ldots,n}$ ($\overline{W}^{\text{axial vector (II)}}_{1,\ldots,n}$) means the same, only the whole linear combination transforms as a pseudoscalar (axial vector). For example

$$\overline{W}^{\text{pseudoscalar (II)}}_{1,2} = \langle 0|A(1)G(2) + \bar{\psi}(1)\gamma_5\psi(2) + G(1)A(2)|0\rangle$$

$$\overline{W}^{\text{axial vector (II)}}_{1,2} = \langle 0|A(1)A_\nu(2) + \bar{\psi}(1)i\gamma_\nu\gamma_5\psi(2) + A_\nu(1)A(2)|0\rangle$$

(b) $\overline{W}^{\text{scalar (IV)}}_{1,\ldots,n}$: the vacuum expectation value from the linear combination of all products of n coefficients of expansion (3.2) with no more than *four* Majorana spinors θ_α and with the whole linear combination transforming as a scalar. For example

$$\overline{W}^{\text{scalar (IV)}}_{1,2} = \langle 0|A(1)D(2) + 2[F(1)F(2) + G(1)G(2) + A_\nu(1)A_\nu(2)] + D(1)A(2) - 2[\bar{\psi}(1)\chi(2) + \bar{\chi}(1)\psi(2)]|0\rangle$$

Now we shall see how from relations (3.4) we can show CPT invariance for Wightman's theory of scalar superfields. It is well known that if we assume relativistic invariance and locality of ordinary fields we can prove CPT invariance of normal Wightman's field theory, where for CPT transformation we can write (for scalar fields)

$$\langle 0|\varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_n(x_n)|0\rangle = \langle 0|\varphi_n(-x_n)\cdots\varphi_2(-x_2)\varphi_1(-x_2)|0\rangle$$

With our symbolic it can be expressed as

$$\begin{aligned}\overline{W}^{\text{scalar (II)}}_{1,\ldots,n} &= \overline{W}^{\text{scalar (II)}}_{-n,\ldots,-1}\\ \overline{W}^{\text{pseudoscalar (II)}}_{1,\ldots,n} &= \overline{W}^{\text{pseudoscalar (II)}}_{-n,\ldots,-1}\\ \overline{W}^{\text{axial vector (II)}}_{1,\ldots,n} &= -\overline{W}^{\text{axial vector (II)}}_{-n,\ldots,-1}\\ \overline{W}^{\text{scalar (IV)}}_{1,\ldots,n} &= \overline{W}^{\text{scalar (IV)}}_{-n,\ldots,-1}\end{aligned} \tag{3.5}$$

We consider the following identities for bilinears $\bar{\theta}0\theta$:

$$(\bar{\theta}0\theta)_{\text{CPT}} = \epsilon(0)\bar{\theta}0\theta$$

where

$$0 \equiv \{\mathbb{1}, \gamma^5, i\gamma^\nu\gamma^5\}$$

$$\epsilon(0) \equiv \{1, 1, -1\}$$

For "Wightman's superfunction" the CPT transformation is

$$\text{CPT}\, W_{\phi_1(x_1,\theta)\cdots\phi_n(x_n,\theta)}(\text{CPT})^{-1} = W_{\phi_n(-x_n,\theta_{\text{CPT}})\cdots\phi_1(-x_1,\theta_{\text{CPT}})} \tag{3.6}$$

If we use in (3.6) relation (3.4) we get

$$W_{\phi_n(-x_n,\theta_{\text{CPT}})\cdots\phi_1(-x_1,\theta_{\text{CPT}})} = \langle 0|A(-n)\cdots A(-1)|0\rangle + \tfrac{1}{4}\bar{\theta}\theta\,\overline{W}^{\text{scalar (II)}}_{-n,\ldots,-1} + \tfrac{1}{4}\bar{\theta}\gamma_5\theta\overline{W}^{\text{pseudoscalar (II)}}_{-n,\ldots,-1} - \tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta\,\overline{W}^{\text{axial vector (II)}}_{-n,\ldots,-1} + \tfrac{1}{32}(\bar{\theta}\theta)^2\,\overline{W}^{\text{scalar (IV)}}_{-n,\ldots,-1} \tag{3.7}$$

If we put (3.5) into (3.7) we obtain

$$W_{\phi_n(-x_n,\theta_{\text{CPT}})\cdots\phi_1(-x_1,\theta_{\text{CPT}})} = W_{\phi_1(x_1,\theta)\cdots\phi_n(x_n,\theta)}$$

So we proved from CPT invariance of Wightman's field theory in R_4 the CPT invariance of "Wightman's superfunctions."

4. *Superfields and the Connection between Spin and Statistics*

Our aim is to shed some light on the further basic result of normal axiomatic field theory—the connection between spin and statistics, now for the case of local quantized superfields.

We shall say that the system of superfields fulfils the normal connection between spin and statistics if the tensor superfields, transforming under one-valued representation of the proper Lorentz group $L_+^\uparrow$, locally commute and the spinor superfields, transforming under two-valued representation of the $L_+^\uparrow$, locally anticommute.

At first we shall recall the axiom of locality for ordinary fields in four dimensions ψ_k (with components ψ_k^α), which can be formulated

$$\psi_k{}^\alpha(x)\psi_l{}^\beta(y) = \sigma_{kl}\psi_l{}^\beta(y)\psi_k{}^\alpha(x), \qquad (x-y)^2 < 0$$

where $\sigma_{kl} = \pm 1$ and is independent of α and β. We note that the supersymmetry transformation induces no Lorentz transformation and we obtain a tensor or spinor superfield by appending a Lorentz index. For simplicity we shall restrict ourselves only to the scalar and the spinor superfield, where for the spinor superfield we can write [in anology to (1.3)]

$$\psi_{\alpha'}(x,\theta) = \bar{\psi}_{\alpha'}(x) + \bar{\psi}_{\alpha}{}^{\alpha}(x)\theta_\alpha + \tfrac{1}{2}\bar{\psi}_{\alpha'}^{[\alpha\beta]}\theta_\beta\theta_\alpha + \tfrac{1}{6}\bar{\psi}_{\alpha'}^{[\alpha\beta\gamma]}(x)\theta_\gamma\theta_\beta\theta_\alpha + \tfrac{1}{24}\bar{\psi}_{\alpha'}^{[\alpha\gamma\beta\sigma]}\theta_\sigma\theta_\gamma\theta_\beta\theta_\alpha \tag{4.1}$$

We know that from the connection between spin and statistics of ordinary fields in four dimensions we have

(a) local commutativity between the spinor field and the boson field
(b) local anticommutativity between spinor fields (4.2)
(c) local commutativity between boson fields

Using conditions (4.2) we can show that we obtain local commutativity for scalar superfields:

$$[\phi(f_1,\theta),\phi(f_2,\theta)] = 0, \qquad \text{supp} f_1(x)\cdot f_2(y) \subset \{(x-y)^2 < 0\}$$

For this purpose we use expansion (2.2):

$$[\phi(f_1,\theta),\phi(f_2,\theta)] = \sum_{j,k=1}^{4} \frac{1}{j!\,k!} \times [\bar{\phi}^{[\alpha_1\cdots\alpha_j]}(f_1)\theta_{\alpha_j}\cdots\theta_{\alpha_1}, \bar{\phi}^{[\beta_1\cdots\beta_k]}(f_2)\theta_{\beta_k}\cdots\theta_{\beta_1}] \quad (4.3)$$

If we use the condition that the product $\theta_{\alpha_1}\theta_{\alpha_2}\cdots\theta_{\alpha_n}$ vanishes for $n > 4$, we get from (4.3) only the following types of commutators between ordinary fields in four dimensions:

$$\text{(i)}\ [\bar{\phi}(f_1),\bar{\phi}(f_2)] = 0 \qquad \text{from (c)}$$

$$[\bar{\phi}^{\alpha_1}(f_1)\theta_{\alpha_1},\bar{\phi}^{\beta_1}(f_2)\theta_{\beta_1}] = -\{\bar{\phi}^{\alpha_1}(f_1),\bar{\phi}^{\beta_1}(f_2)\}\,\theta_{\alpha_1}\theta_{\beta_1} = 0 \qquad \text{from (b)}$$

$$[\bar{\phi}^{[\alpha_1\alpha_2]}(f_1)\theta_{\alpha_2}\theta_{\alpha_1},\bar{\phi}^{[\beta_1,\beta_2]}(f_2)\theta_{\beta_2}\theta_{\beta_1}] = 0 \qquad \text{from (c)}$$

$$\text{(ii)}\ [\bar{\phi}(f_1),\bar{\phi}^{\beta_1}(f_2)\theta_{\beta_1}] = 0 \ \text{from (a)}$$

$$[\bar{\phi}(f_1),\bar{\phi}^{[\beta_1\beta_2]}(f_2)\theta_{\beta_2}\theta_{\beta_1}] = 0 \qquad \text{from (c)}$$

$$[\bar{\phi}(f_1),\bar{\phi}^{[\beta_1\beta_2\beta_3]}(f_2)\theta_{\beta_3}\theta_{\beta_2}\theta_{\beta_1}] = 0 \qquad \text{from (a)}$$

$$[\bar{\phi}(f_1),\bar{\phi}^{[\beta_1\beta_2\beta_3\beta_4]}(f_2)\theta_{\beta_4}\theta_{\beta_3}\theta_{\beta_2}\theta_{\beta_1}] = 0 \qquad \text{from (c)}$$

$$\text{(iii)}\ [\bar{\phi}^{\alpha_1}(f_1)\theta_{\alpha_1},\bar{\phi}^{[\beta_1\beta_2]}(f_2)\theta_{\beta_2\beta_1}] = 0 \qquad \text{from (a)}$$

$$[\bar{\phi}^{\alpha_1}(f_1)\theta_{\alpha_1},\bar{\phi}^{[\beta_1\beta_2\beta_3]}(f_2)\theta_{\beta_3}\theta_{\beta_2}\theta_{\beta_1}] = 0 \qquad \text{from (b)}$$

It is remarkable that the local commutativity for scalar superfields is independent of commutativity or anticommutativity between θ_α and the Fermi fields.

A different situation arises for spinor superfields, because the anticommutativity between θ_α and the Fermi fields must hold strictly as we shall see below. For anticommutators between spinor superfields we obtain

$$\{\psi_{\alpha'}(f_1,\theta),\psi_{\beta'}(f_2,\theta)\}_+ = \sum_{j,k=1}^{4} \frac{1}{j!\,k!} \times \{\bar{\psi}_{\alpha'}^{[\alpha_1\cdots\alpha_j]}(f_1)\theta_{\alpha_j}\cdots\theta_{\alpha_1}, \bar{\psi}_{\beta'}^{[\beta_1\cdots\beta_k]}\theta_{\beta_k}\cdots\theta_{\beta_1}\}$$

and analogously we get only following types of anticommutators between ordinary fields:

$$\text{(j)}\ \{\bar{\psi}_{\alpha'}(f_1),\bar{\psi}_{\beta'}(f_2)\} = 0 \qquad \text{from (b)}$$

$$\{\bar{\psi}_{\alpha'}^{\alpha_1}(f_1)\theta_{\alpha_1},\bar{\psi}_{\beta'}{}^{\beta_1}(f_2)\theta_{\beta_1}\} = [\bar{\psi}_{\alpha'}{}^{\alpha_1}(f_1),\bar{\psi}_{\beta'}{}^{\beta_1}(f_2)]\,\theta_{\alpha_1}\theta_{\beta_1} = 0 \qquad \text{from (c)}$$

$$\{\bar{\psi}_{\alpha'}^{[\alpha_1\alpha_2]}(f_1)\theta_{\alpha_2}\theta_{\alpha_1},\bar{\psi}_{\beta'}^{[\beta_1\beta_2]}(f_2)\theta_{\beta_2}\theta_{\beta_1}\} = 0 \qquad \text{from (b)}$$

$$\text{(jj)}\ \{\bar{\psi}_{\alpha'}(f_1),\bar{\psi}_{\beta'}{}^{\beta_1}(f_2)\theta_{\beta_1}\} = [\bar{\psi}_{\alpha}(f_1),\bar{\psi}_{\beta'}{}^{\beta_1}(f_2)]\,\theta_{\beta_1} = 0 \qquad \text{from (a)}$$

If we use commutativity between θ_α and the Fermi field as we can make it in the case of scalar superfields, we obtain an anomalous connection between spin and statistics for ordinary fields:

$$\{\bar{\psi}_{\alpha'}(f_1), \bar{\psi}_{\beta'}^{[\beta_1\beta_2]}(f_2)\theta_{\beta_2}\theta_{\beta_1}\} = 0 \qquad \text{from (b)}$$

$$\{\bar{\psi}_{\alpha'}(f_1), \bar{\psi}_{\beta'}^{[\beta_1\beta_2\beta_3]}(f_2)\theta_{\beta_3}\theta_{\beta_2}\theta_{\beta_1}\} = 0 \qquad \text{from (a)}$$

$$\{\bar{\psi}_{\alpha'}(f_1), \bar{\psi}_{\beta'}^{[\beta_1\beta_2\beta_3\beta_4]}(f_2)\theta_{\beta_4}\theta_{\beta_3}\theta_{\beta_2}\theta_{\beta_1}\} = 0 \qquad \text{from (b)}$$

$$\text{(jjj)}\ \{\bar{\psi}_{\alpha'}{}^{\alpha_1}(f_1)\theta_{\alpha_1}, \bar{\psi}_{\beta'}^{[\beta_1\beta_2]}(f_2)\theta_{\beta_2}\theta_{\beta_1}\} = 0 \qquad \text{from (a)}$$

$$\{\bar{\psi}_{\alpha'}{}^{\alpha_1}(f_1)\theta_{\alpha_1}, \bar{\psi}_{\beta'}^{[\beta_1\beta_2\beta_3]}(f_2)\theta_{\beta_3}\theta_{\beta_2}\theta_{\beta_1}\} = 0 \qquad \text{from (c)}$$

Also for the commutator between the scalar and the spinor superfield we obtain in the same way

$$[\phi(f_1, \theta), \psi_{\alpha'}(f_2, \theta)] = 0, \qquad \operatorname{supp} f_1(x) \cdot f_2(y) \subset \{(x-y)^2 < 0\}$$

Concluding: from the normal connection between spin and statistics of ordinary fields in four dimensions, we get

$$\left.\begin{aligned} [\phi(f_1, \theta), \psi_{\alpha'}(f_2, \theta)] &= 0 \\ \{\psi_{\alpha'}(f_1, \theta), \psi_{\beta'}(f_2, \theta)\} &= 0 \\ [\phi(f_1, \theta), \phi(f_2, \theta)] &= 0 \end{aligned}\right\} \quad \operatorname{supp} f_1(x) \cdot f_2(y) \subset \{(x-y)^2 < 0\}$$

That means that the superfields fulfill the normal connection between spin and statistics.

Now we shall prove the general theorem on the connection between spin and statistics for local quantized superfields, using "Wightman's superfunction.' First we shall state some basic assumptions. We use the notation $D(j/2, k/2)$ for irreducible representation (IR) of the group $SL(2, C)$, where j and k are non-negative integers. If the superfield $\psi(x, \theta)$ transforms under the representation $D(j/2, k/2)$, then $\psi^*(x, \theta)$ transforms under the adjoint representation $D(k/2, j/2)$. We know that "Wightman's superfunction" is the sum of ordinary Wightman's functions, and so we can formally write in symbolic form

$$W_{\Phi_1(x_1,\theta)\Phi_2(x_2,\theta)} = \langle 0|\bar{\phi}(1)\bar{\phi}(2)|0\rangle + \bar{W}_{1,2}^{\text{tensor (II)}}\theta_\beta\theta_\alpha + \bar{W}_{1,2}^{\text{tensor (IV)}}\theta_\sigma\theta_\gamma\theta_\beta\theta_\alpha \tag{4.4}$$

$$W_{\psi_{1\alpha'}(x_1,\theta)\psi_{2\beta'}(x_2,\theta)} = \langle 0|\bar{\psi}_{\alpha'}(1)\bar{\psi}_{\beta'}(2)|0\rangle + \bar{W}_{1,2}^{\text{spinor (II)}}\theta_\beta\theta_\alpha + \bar{W}_{1,2}^{\text{spinor (IV)}}\theta_\sigma\theta_\gamma\theta_\beta\theta_\alpha$$

where only members with even numbers of ordinary spinor fields do not vanish in vacuum expectation value.

It is remarkable that the vacuum expectation value of odd-numbered spinor superfields is not evidently zero. The reason for that is that supersymmetry is Fermi–Bose symmetry mixing scalar with spinor fields and so we obtain for example, in the product of three spinor superfields, the member $\bar{\psi}_{\alpha'}(x_1)\bar{\psi}_{\beta'}{}^{\beta_1}(x_2)\bar{\psi}_{\gamma'}(x_3)\theta_{\beta_1}$ and the vacuum expectation value $\langle 0|\bar{\psi}_{\alpha'}(x_1)\bar{\psi}_{\beta'}{}^{\beta_1}(x_2)\bar{\psi}_{\gamma'}(x_3)|0\rangle \neq 0$.

Because the relations (4.4) are valid, we can transfer all characteristics from normal Wightman's theory to theory of "Wightman's superfunctions" as well as the Bargman-Hall-Wightman theorem which we need for our proof (Streater and Wightman, 1964). We need also the following lemma:

> *Lemma.* If $\psi(x,\theta)|0\rangle = 0$, where $\psi(x,\theta)$ is a local superfield, then $\psi(x,\theta) \equiv 0$. This is fulfilled for ordinary fields, and because the general superfield is exactly equivalent to a set of ordinary fields, this lemma holds also for superfields.

Now we can formulate the theorem on the connection between spin and statistics for superfields:

> *Theorem.* Let $\psi(x,\theta)$ be a complex superfield, which transforms under arbitrary (IR) $D(j/2, k/2)$ of the group $SL(2, C)$. If
> $$\psi_\alpha(x,\theta)\psi_\alpha{}^*(y,\theta) = -(-1)^{j+k}\psi_\alpha{}^*(y,\theta)\psi_\alpha(x,\theta) \tag{4.5}$$
> holds for $(x-y)^2 = \xi^2 < 0$, then $\psi_\alpha(x) \equiv 0$.

Proof. Because we work only with one component α, we shall omit index α. We designate "Wightman's superfunction" as

$$F_1(x-y,\theta) = \langle 0|\psi(x,\theta)\psi^*(y,\theta)|0\rangle$$

and

$$F_2(x-y,\theta) = \langle 0|\psi^*(x,\theta)\psi(y,\theta)|0\rangle$$

From (4.5) we get for $\xi^2 < 0$

$$F_1(\xi,\theta) + (-1)^{j+k}F_2(-\xi,\theta) = 0 \tag{4.6}$$

where $\xi = (x-y) \subset R_4$. We can do an analytic continuation of the equation (4.6) to an extended tube T', where $T' \equiv \underset{\Lambda\epsilon L_+(C)}{\{U\Lambda T^\pm} \,|\, \zeta\epsilon T^\pm = R_4 + i\bar{V}_\pm, L_+(C)-$ the group of complex proper Lorentz transformation$\}$. Because the "Wightman's superfunction" is the sum of covariant Wightman's functions under $L_+(C)$, it is evidently also covariant under $L_+(C)$, especially under transformation of the space-time discrete symmetry. In this case the following relation is fulfilled:

$$F_2(-\zeta,\theta) = (-1)^{j+k}F_2(\zeta,\theta) \tag{4.7}$$

If we put (4.7) into (4.6) we get

$$F_1(\zeta,\theta) + F_2(\zeta,\theta) = 0$$

and in the limit case $\bar{V}_- \ni Im\zeta \to -0$ we obtain for real x and y

$$\langle 0|\psi(x,\theta)\psi^*(y,\theta)|0\rangle + \langle 0|\psi^*(x,\theta)\psi(y,\theta)|0\rangle = 0 \tag{4.8}$$

After multiplying equation (4.8) by testing functions, we have

$$\| \psi^*(f, \theta) |0\rangle \|^2 + \| \psi(f, \theta) |0\rangle \|^2 = 0$$

and from the lemma we obtain $\psi(x, \theta) \equiv 0$.

So we have proved that anomalous commutation or anticommutation rules for superfields are forbidden.[1]

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[1] It is also evident from relation (4.5) using the theorem on the connection between spin and statistics for ordinary fields.